# Steady-State Analysis of Circuits with Multiple Adaptive Grids

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***Abstract*— This paper explores an effective adaptive algorithm to obtain the steady-state response of circuits with periodic excitation. This algorithm offers an alternative to conventional steady-state simulation techniques such as harmonic balance or shooting method. One of the contributions of this paper is the generic formulation of nonlinear circuit equations in terms of transformation matrices. This algorithm is especially advantageous to circuits showing high nonlinearity. In the proposed method, each circuit variable is represented by a linear combination of a set of adaptive basis functions (ABF). Waveform samples are obtained by multiplying a vector of coefficients times a transformation matrix. Furthermore, in order to improve the output resolution, the positions of the basis functions for each variable are adaptively controlled. In other words, we start the analysis with a uniform grid and increase the resolution where needed by adjusting the position of the grid points independently for each variable. The least squares method is used to find the set of coefficients that minimizes the error function defined by the formulation. We present for the first time simulation results with different grids in each state variable.**

***Index Terms*— Steady-State Analysis, Adaptive Algorithm, ABF, Least Squares Method, Multiple Adaptive Grids**

## I. Introduction

The steady-state response is of primary concern in many nonlinear circuits. RF oscillators and amplifiers are some examples. Several techniques exist to find the steady-state response [1], [2] such as harmonic balance (HB) [3], [5] or the shooting method [6], [7]. The HB technique is associated with the Fourier expansion for each state variable. In this technique, the coefficients of the Fourier Series are adjusted such that they satisfy the circuit equations. Thus a large number of variables must be optimized leading to great density in Jacobian matrix when circuits show high nonlinearity.

Only a few methods have been proposed to analyze circuits with basis functions other than sinusoids (*i.e.*, HB). In references [4], [8]–[12] pseudo-wavelet or wavelet basis functions were used for time domain analysis of circuits. Publications [4], [11], [12] are particularly intended for steady-state analysis. The method in [11] is for steady state analysis but variable resolution is not implemented there. Another approach to represent the state variables is the use of adaptive basis functions [13] instead of wavelets. This approach shares most of the advantages of wavelets (variable resolution, local support that originates sparse matrices). ABF can be located in terms of a non-uniform grid. In [13] a Galerkin approach was used to find the ABF coefficients of variables. However, it is not clear in [13] if a different grid was used for each state variable. One disadvantage of using the Galerkin approach for general nonlinear systems is that the integrals must be solved numerically, which results in the calculation of many intermediate points in every interval.

This paper is organized as follows: in Section II we present the mathematical background and formulate the equations and an application example is presented in Section III followed by the conclusions.

## II. Equation Formulation

### A. Adaptive Basis Functions

In the proposed adaptive algorithm, the unknowns in the circuit equations are the coefficients that multiply the ABF. A state variable $v(t)$ is approximated by basis function series

$$v(t) \simeq \bar{v}(t) = \sum_{i=0}^{N} \varphi_i(t) \cdot \hat{v}_i \tag{1}$$

where $\bar{v}(t)$ is the approximation form of $v(t)$ and $\varphi_i(t)$ represents an ABF. The ABFs have two forms: $\varphi_{2j}(t)$ and $\varphi_{2j+1}(t)$ as illustrated in Fig. 1. Initially, the basis functions are located in $n$ equally spaced points in the simulation period ($T$). For each interval defined by these points we consider 3 sample points so the total number of grid points is $3n$. The $i$th element in a coefficient vector ($\hat{V}$) is denoted $\hat{v}_i$. Thus $\hat{V} = [\hat{v}_1, \hat{v}_2, ...\hat{v}_i, ...\hat{v}_{2n}]^T$.

### B. Transformation Matrices

Consider a vector $U$ that consists in the concatenation of vectors of sample points with the nodal voltages,

$$U = \left[\ U_1, U_2, \ldots, U_i, \ldots, U_m\ \right]^T\ , \tag{2}$$

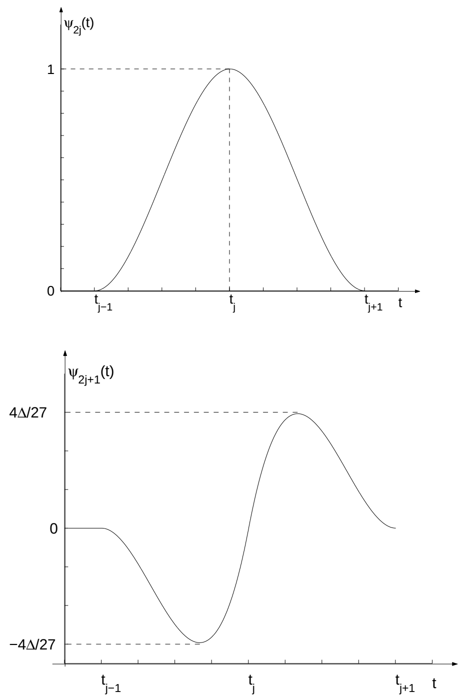


Fig. 1. Adaptive basis function

where $U_i$ denotes a vector with the samples of the voltage at Node $i$ at a sequence of points $t_j$ as shown in Eq. (3).

$$U_i = \left[ \ U_i(t_0), U_i(t_1), \ldots, U_i(t_j), \ldots \ \right]^T \tag{3}$$

It is possible to define the transformation matrices $B$ and $B_p$ that relate the ABF coefficients for Node $i$ in vector $\hat{U}_i$ with the samples in $U_i$ as follows:

$$U_i = B\hat{U}_i \ , \ \dot{U}_i = B_p\hat{U}_i. \tag{4}$$

We can now define the vector with ABF coefficients for all nodal variables,

$$\hat{U} = [\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_i, \ldots, \hat{U}_m]^T$$

The aim of using transformation matrices is to convert the coefficient vector $\hat{U}_i$ into the respective values of time samples $U_i$ and their derivatives $\dot{U}_i$. Matrices $B$ and $B_p$ are sparse because, in the worst case, only four elements per row are different from zero. For instance, considering a case with $n = 10$; $2n$ basis functions are used and $3n$ sample points are used as discussed earlier in the paper. The $B$ matrix turns out to be a rectangular matrix of $2n \times 3n$ (same to $B_p$). The nonzero map of $B$ with equally spaced grid points is shown in Fig. 2.

### C. Expansion of Circuit Equations with Uniform Grid

Consider the nodal equation formulation of a circuit:

$$Gu(t) + C\dot{u}(t) + I(u(t)) + \dot{Q}(u(t)) = S(t) \tag{5}$$

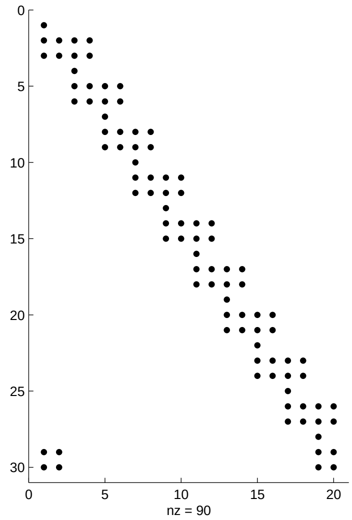


Fig. 2. Nonzero structure of $B$ with $n = 10$

where $G$ and $C$ are square matrices that represent the linear conductance and charge terms. The vector functions $I(u(t))$ and $\dot{Q}(u(t))$ represent the contributions of the nonlinear algebraic and charge terms, respectively. The vector of the nodal voltages and selected currents is denoted $u(t)$ and $S(t)$ is a vector of independent sources . The functions in $u(t)$ and $S(t)$ satisfy periodical boundary conditions:

$$u(t+T) = u(t) \ , \ S(t+T) = S(t).$$

It is convenient to separate linear and nonlinear terms in Eq. (5) for more efficient numerical solutions. Considering the nonlinear terms $I(u)$ and $Q(u)$, Eq. (5) is reformulated as

$$\left\{ \begin{array}{rcl} Gu(t) + C\dot{u}(t) + \bar{M}\dot{q} + I(u(t)) & = & S(t) \\ -q + Q(u(t)) & = & 0 \end{array} \right. , \tag{6}$$

where $\bar{M}$ denote a mapping (or incidence) matrix and $q$ is an auxiliary vector with all charge terms is the circuit. A key characteristic of Eq. (6) is that the derivative with respect to time is applied to $q$ instead of the nonlinear charge function $Q(u(t))$. This makes possible the calculation of time derivatives from the ABF coefficients of $q$ using the $B_p$ matrix described before. Eq. (6) is solved using Newton-Raphson method. The nonlinear terms at the next iteration $I(u^{j+1})$, $Q(u^{j+1})$ are estimated as follows,

$$\begin{array}{rcl} I(u^{j+1}) & \simeq & I(u^j) + J_I(u^{j+1} - u^j) \\ Q(u^{j+1}) & \simeq & Q(u^j) + J_Q(u^{j+1} - u^j) \ , \end{array}$$

where $J_I$ and $J_Q$ are Jacobian matrix corresponding to $I(u_j)$ and $Q(u_j)$, respectively. Substituting the nonlinear terms in Eq. (6) by the Newton approximations and rearranging we obtain

$$\underbrace{\begin{bmatrix} G+J_I^j & 0 \\ J_Q^j & -M_q \end{bmatrix}}_{G'} \underbrace{\begin{bmatrix} u^{j+1} \\ q^{j+1} \end{bmatrix}}_{u'} + \underbrace{\begin{bmatrix} C & \bar{M} \\ 0 & 0 \end{bmatrix}}_{C'} \underbrace{\begin{bmatrix} \dot{u}^{j+1} \\ \dot{q}^{j+1} \end{bmatrix}}_{\dot{u}'}$$

$$= \underbrace{\begin{bmatrix} S - I(u^j) + J_I^j u^j \\ -Q(u^j) + J_Q^j q^j \end{bmatrix}}_{\mathbf{S}'}, \quad (7)$$

where $M_q$ is an identity matrix matching the dimension of $q$. In short form, Eq. (7) is similar to the equation of a linear circuit,

$$G'u' + C'\dot{u}' = \mathbf{S}' \; . \quad (8)$$

Now we derive the equations for a linear circuit with uniform grid. Generality is not lost because if the circuit is nonlinear, an equivalent system of equations (Eq. 8) can be derived at each Newton iteration. Consider the differential equation describing a linear circuit,

$$GU + C\dot{U} = \mathbf{S} \; , \quad (9)$$

and rewrite Eq. (9) as a function of $\hat{U}$, where $\otimes$ denotes Kronecker product:

$$\underbrace{(G \otimes B + C \otimes B_p)}_{A} \hat{U} = \underbrace{\mathbf{S}}_{\mathbf{b}} \; . \quad (10)$$

The System of Equations (10) is over-determined because the total number of equations here is larger than the unknowns. The least squares method is used in this paper. As a consequence, the unknowns $\hat{U}$ are computed as follows:

$$A^T A \cdot \hat{U} = A^T \mathbf{b}. \quad (11)$$

### *D. Expansion of Circuit Equations with Non-Uniform Grid*

Now consider that the grid points are not equally spaced. For all nodal voltages in a circuit, transformation matrices ($B$ or $B_p$) have to be updated corresponding to each state variable:

$$\begin{bmatrix} g_{11}B_1 + c_{11}B_{p1} & \cdots & g_{1m}B_m + c_{1m}B_{pm} \\ g_{21}B_1 + c_{21}B_{p1} & \cdots & g_{2m}B_m + c_{2m}B_{pm} \\ \vdots & & \\ g_{m1}B_1 + c_{m1}B_{p1} & \cdots & g_{mm}B_m + c_{mm}B_{pm} \end{bmatrix}$$

$$\times \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \\ \cdots \\ \hat{U}_m \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_m \end{bmatrix} . \quad (12)$$

The system is first solved using uniform grids for all nodal voltages. The results of a given solution are used to refine the positions of the grid points for the next iteration (*i.e.* non-uniform grids). Then we project all grids onto a reference grid and Eq. (12) is evaluated considering all points of the reference grid. It is not possible to know in advance how many equations will be considered with nonuniform grids. But in general, the number of equations will be greater than the one with uniform grids. The strategy used to refine the grid positions is described next.

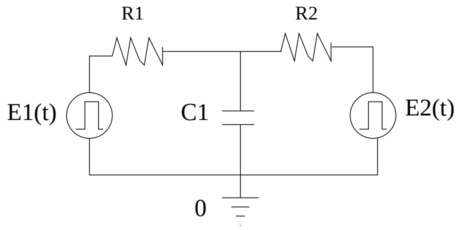

Fig. 3. RC circuit

### *E. Grid Adaptation*

Cook and Duncan [14] explore a grid adaptation technique for PDEs with one variable when the solutions show a steep trend. The locations of discrete grid points where the ABFs are located is adapted by evenly distributing area residuals as defined in [14]. A ‘drastic activity’ region on a waveform requires a higher density of ABF grid points than a ‘low activity’ region. We apply a similar approach independently for each state variable in the circuit to obtain multiple adaptive grids. When the new grids are available we recalculate the individual transformation matrices ($B$ and $B_p$) for each variable and reformulate Eq. (12). Furthermore, if the circuit is nonlinear, the coefficient vector $\hat{U}$ must also be updated for the new grids and saved as the next guess for the Newton method. The iterations end when the area residuals for all state variables are evenly distributed [15].

### *F. Summary*

We construct generic circuit equations in terms of matrices $B$ and $B_p$, associated with their coefficient vectors $\hat{U}$, and solve them by the least squares method. The circuit waveforms are first approximated with uniform grids. Next, the solution is refined using non-uniform grids by implementing the grid adaptation strategy and recalculating the circuit solutions with higher accuracy without increasing the total number of unknowns (*i.e.*, the ABF coefficients).

## III. Application Example

The RC circuit shown Fig. 3 is used to test the algorithm. The excitations $E_1$ and $E_2$ are identical square waves with a period of 1 $\mu$s and a phase shift of 0.25 $\mu$s. Since square waves imply many harmonics and sharp variations, the time domain method is a good choice for this circuit.

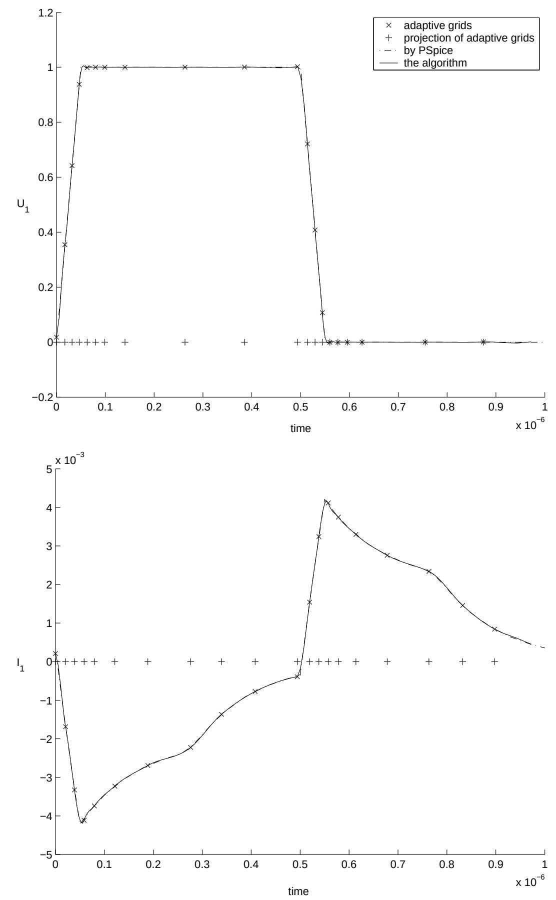


Fig. 4. Results with non-uniform grids compared to PSpice (20 intervals)

This linear circuit requires 5 state variables (*i.e.*, $m = 5$). The algorithm starts with 20 uniform intervals in a period followed by the adaptation procedure. Thus, the vector $\hat{U}_i$ has 40 elements. Both $B_i$ and $B_{pi}$ (there is one pair per state variable) have 20 columns and the the number of rows varies according to the adaptation.

The computation ends after 15 iterations. Some of the results are shown in Fig. 4. The algorithm was implemented in Matlab 7. The total CPU time on a Pentium 4 computer was 10.33 s. It can be observed in Fig. 4 that the grids are different for each variable and that the grids are denser in the regions where the waveforms present drastic activity.

If 10 non-uniform intervals are used, the nonzero map of matrix $A^T A$ in Eq. (11) has a density of 27.84 percent. For this RC small circuit, the matrix $A^T A$ is not very sparse. However, for a large circuit, the density will be decreased significantly since the matrices $G$ and $C$ are sparser.

## IV. Conclusions

In this paper we have developed a method to analyze the steady state of circuit using multiple adaptive grids. We have partially tested the method with the analysis of a linear circuit. We have effectively applied ABF to approximate the waveform of each state variable. The generic circuit equations were formulated by transformation matrices and solved by the least squares method. The method presented in this paper provides a means to increase the accuracy of circuit solutions without increasing the number of unknowns. For nonlinear circuits this requires another set of Newton iterations (but with a good initial guess) to update the ABF coefficients. Further development and testing of the method for nonlinear circuits will be the subject of future research. To our knowledge, we have presented for the first time simulation results with different grids in each state variable.

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